Double limit 2

Find the value of $a \in \mathbb{R}$ such that the function is continuous throughout its domain.

$$f(x,y) = \begin{cases} \frac{x^3 - x(y+1)^2}{x^2+(y+1)^2} & \text{if } (x,y) \neq (0,-1) \\ a & \text{if } (x,y) = (0,-1) \end{cases}$$

Solution

The function only presents problems at $x, y = 0, -1$. Therefore, if it is continuous at that point, it is continuous throughout its domain. The function exists at the point, so the only remaining condition is that the limit of the function also exists at the point and equals a.

$$\lim_{(x,y)\to(0,-1)} \frac{x^3 - x(y+1)^2}{x^2+(y+1)^2} = \lim_{(x,y)\to(0,-1)} \frac{x(x^2-(y+1)^2)}{x^2+(y+1)^2} = \lim_{(x,y)\to(0,-1)} x\left(\frac{x^2}{x^2+(y+1)^2} - \frac{(y+1)^2}{x^2+(y+1)^2}\right)$$

Both terms in parentheses are bounded

$$0 \leq \frac{x^2}{x^2+(y+1)^2} \leq 1$$

$$0 \leq \frac{(y+1)^2}{x^2+(y+1)^2} \leq 1$$

Therefore, the difference between them is also bounded, with a maximum possible value of 1 and a minimum of -1. The result is then a term that tends to 0 and another term that is bounded. **By the theorem of the product of an infinitesimal and a bounded function, the result of the limit is 0 and therefore a must be 0 for the function to be continuous.**